Differential equation using integrating factor

Solve the following differential equation:

$$2xy\ln(y)\,dx + (x^2 + y^2\sqrt{y^2+1})\,dy = 0$$

Solution

We see that it is not exact

$$P'_y = 2x\ln(y) + 2x$$
$$Q'_x = 2x$$
$$\Rightarrow P'_y \neq Q'_x$$

We find the integrating factor

$$\varphi(y) = e^{\int \frac{Q'_x(x,y) - P'_y(x,y)}{P(x,y)}\,dy}$$

Where $\frac{Q'_x(x,y)-P'_y(x,y)}{P(x,y)} = \frac{1}{y}$

$$e^{-\int \frac{1}{y}\,dy} = e^{-\ln(y)} = e^{\ln(y^{-1})} = y^{-1}$$

The integrating factor depends solely on y. We multiply the entire equation by the integrating factor

$$y^{-1}2xy\ln(y)\,dx + y^{-1}(x^2 + y^2\sqrt{y^2+1})\,dy = 0$$

$$2x\ln(y)\,dx + \left[\frac{x^2}{y} + y\sqrt{y^2+1}\right]\,dy = 0$$

Let's check that this equation is exact

$$P'_y = \frac{2x}{y}$$
$$Q'_x = \frac{2x}{y}$$
$$\Rightarrow P'_y = Q'_x$$

We solve the exact differential equation.
We look for $U(x,y)$

$$U(x,y) = \int P(x,y)\,dx = \int 2x\ln(y)\,dx = x^2\ln(y) + C(y)$$

We derive with respect to y and equate it to $Q(x,y)$

$$U'_y = \frac{x^2}{y} + C'(y) = \frac{x^2}{y} + y\sqrt{y^2+1}$$
$$C'(y) = y\sqrt{y^2+1}$$

Integrating

$$\int y\sqrt{y^2+1}dy$$

Using substitution method:

$$u = y^2 + 1$$
$$du = 2ydy$$
$$du/2 = ydy$$

$$\frac{1}{2}\int \sqrt{u}du = \frac{1}{2}u^{3/2}\frac{2}{3} + C = \frac{(y^2+1)^{3/2}}{3} + C$$

$$C(y) = \frac{1}{3}(y^2+1)^{3/2} + C$$

Replacing, we obtain that the solution is:

$$x^2\ln(y) + \frac{1}{3}(y^2+1)^{3/2} + C = 0$$